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# Zeros of Padé Error Functions for Functions with Smooth Maclaurin Coefficients

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We deal with functions $f(z) := \sum_{n=0}^{\infty} a_n z^n$ whose coefficients satisfy Lubinsky's smoothness condition, namely, $a_{j+1} \cdot a_{j-1}/a_j^2 \to \eta$ *as* $j \to \infty$, $\eta \neq \infty$. In the present paper, theorems concerning the asymptotic behaviour of the normalized (in an appropriate way) Padé error functions $(f - \pi_{n,m})$ as $n \to \infty$, $m$-fixed, are provided. As a consequence, results concerning the number of the zeros and of their limiting distribution are deduced. 

## 1. Introduction and Main Results

Let

$$f(z) := \sum_{j=0}^{\infty} a_j z^j \tag{1.1}$$

be a function with $a_j \neq 0$ for all nonnegative integers $j$, $(j \in \mathbf{N})$ large. We set

$$\eta_j := a_{j+1} \cdot a_{j-1}/a_j^2, \qquad j = j_0, j_1, \dots .$$

The basic assumption throughout the present work is that

$$\eta_j \to \eta, \qquad \text{as} \quad j \to \infty. \tag{1.2}$$

This kind of convergence has been introduced and studied by D. Lubinsky in [7], where important theorems resulting from (1.2) with respect to the asymptotic of Toeplitz determinants and uniform convergence of the $m$th row of the table of classical Padé approximants to $f$ are proved. Therefore, in what follows, condition (1.2) will be called "Lubinsky's smoothness condition".

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Let $\rho(f)$ be the radius of convergence of the power series (1.1). We notice that under Lubinsky's smoothness condition (1.1) represents an entire function, if $|\eta| < 1$; the radius of convergence $\rho(f)$ is zero, if $|\eta| > 1$. If $|\eta| = 1$, then (1.1) may have a positive or a zero radius of convergence.

Further, we assume that the numbers $\eta_n$ tend to $\eta$ smoothly enough, namely: there exist complex numbers $\{c_i\}_{i=1}^{\infty}$ with $c_1 \neq 0$ such that for each $N$, $N \in \mathbf{N}$, $N > 1$, we have the representation

$$\eta_n = \eta \cdot \left\{1 + c_1/n + \sum_{i=2}^{N} c_i/n^i + o(n^{-N})\right\}. \tag{1.3}$$

Introduce the function $H_\eta(z)$ with

$$H_\eta(z) := \sum_{j=0}^{\infty} \eta^{j(j+1)/2} z^j.$$

It is clear that for $|\eta| < 1$, $H_\eta(z)$ is an entire function. If $|\eta| = 1$, then $H_\eta(z)$ is holomorphic in the unit disk; in the case when $|\eta| > 1$, the radius of convergence is zero.

Notice that $H_\eta(z) = h(\eta z)$, where $h(z)$ is the partial theta function. Its properties (natural boundary, domains omitting zeros etc.) have been studied in [8].

Let now $m$ be a nonnegative integer. In our further considerations, we will assume that $m$ is fixed.

Further, we assume that the power series (1.1) does not represent a rational function with a number of finite poles, not more than $m$ (we write $f \notin \mathscr{R}_m$).

For each $n$, $n \in \mathbf{N}$, let $\pi_{n,m}$ $(= \pi_{n,m}(f))$ be the Padé approximant to the function $f$ of order $(n, m)$. We set

$$\pi_{n,m} = P_{n,m}/Q_{n,m},$$

where $Q_{n,m}(0) = 1$ and both polynomials $P_{n,m}$ and $Q_{n,m}$ do not have a common divisor.

Let $D(n, m) = det\{a_{n-j+k}\}_{j,k=1}^{m}$ be Toeplitz's determinant formed from the Maclaurin coefficients of (1.1). Under our basic assumption concerning nonrationality of $f$, it is true that the inequality $D(n, m) \neq 0$ holds for an infinite sequence of positive integers $n$ (see, for instance, [2] and [3]). Denote by $\Lambda$ the sequence of those positive integers for which $deg\, Q_{n,m} = m$; as it is known (see [2], [3]) $\Lambda$ is infinite (recall that that $f \notin \mathscr{R}_m$). For any $n \in \mathbf{N}$ the equality $\pi_{n,m} \equiv \pi_{k(n),m}$, where $k(n) := \max\{k,\ k \leqslant n,\ k \in \Lambda\}$ is valid. For $n \in \Lambda$ there holds (see [9])

$$(f \cdot Q_{n,m} - P_{n,m})(z) = z^{n+m+1} \cdot (-1)^m \cdot D(n+1, m+1)/D(n, m) + \cdots,$$

and

$$Q_{n,m}(z) = 1 + \cdots + z^m \cdot (-1)^m D(n+1, m)/D(n, m).$$

Recalling now the structure of Padé's table corresponding to $f$, we assume without losing the generality that the last formulas hold for every $n \in \mathbf{N}$ starting with number $n_0$.

In what follows, we shall call the difference $f - \pi_{n,m}$ the Padé error function to $f$ of order $(n, m)$.

Recently, the limiting distribution of the zeros of the $m$th row in Padé's table for entire functions with "smooth" Maclaurin coefficients was considered (see [5]). As a consequence, the limiting distribution of the zeros of the sequence of Padé approximants $\pi_{n,m}$ as $n \to \infty$ was characterized. The goal of the present paper is to explore analogous problems with respect to the Padé error functions.

Denote by $S_n(z) = S_n(f, z)$ the $n$th partial sum of the function $f(z)$:

$$S_n(z) := \sum_{j=0}^{n} a_j z^j.$$

We notice that $S_n(z) = \pi_{n,0}(z)$ for every $N \in \mathbf{N}$.

The starting point for the investigations is the following unpublished result by E. B. Saff with describes the limiting behaviour of the differences $f - S_n$ as $n \to \infty$ normalized in an appropriate way.

THEOREM 1. *Set*

$$W_n(u) := (f - S_n)(ua_n/a_{n+1})/a_{n+1}(ua_n/a_{n+1})^{n+1}.$$

*Assume that* (1.2) *holds with* (i): $|\eta| < 1$ *and* (ii): *with* $|\eta| = 1$ *in a way that* $|\eta_n| \leqslant 1$ *for all n large enough. Then, respectively,*

(i)

$$W_n(u) \to H_\eta(u) \tag{1.4i}$$

*uniformly inside in* $\mathbf{C}$ *and*

(ii)

$$W_n(u) \to H_\eta(u) \tag{1.4ii}$$

*uniformly inside* $\{u : |u| < 1\}$.

As usual, "uniformly inside" a given set $M$, $M \in \mathbf{C}$, means uniform convergence on compact subsets of $M$ in the uniform norm.

Let now $m$ be a fixed positive integer. The first result in the present paper refers to functions (1.1) for those $\eta \neq 1$. Following [8], we introduce the polynomials $B_m(u) := B_m(u, q)$, $m \in \mathbb{N}$ fixed, as follows:

$B_o(u) := 1$ and for $m = 1, 2, \ldots$

$$B_m(u) := B_{m-1}(u) - u \cdot q^{m-1} \cdot B_{m-1}(u/q).$$

When $q$ is not a root of unity, then

$$B_m(-u) = \sum_{j=0}^{m} \frac{u^j \prod_{k=1}^{j} (1 - q^{m+1-k})}{\prod_{k=1}^{j} (1 - q^k)};$$

furthermore, $B_m(u) = (1-u)^m$, when $q = 1$.

These polynomials are of importance in the investigation of the distribution of the zeros of Padé error functions $f - \pi_{n,m}$ in the case when the number $\eta$ in (1.2) is not a root of unity.

For $0 < q < 1$, the polynomials $B_m$ (suitably normalized) are orthogonal with respect to a nonnegative weight on the unit circle (see [1]), so that all their zeros lie in $\{z, |z| \leqslant 1\}$. For $q = 1$ results concerning the distribution of the zeros of the polynomials $B_m(u)$, $m = 0, 1, \ldots$ can be found in [8].

For our goal, we introduce an appropriate normalization of the error functions. Set

$$e_{n,m} := \frac{(f - \pi_{n,m})(ua_n/a_{n+1})}{a_{n+1}(ua_n/a_{n+1})^{n+1}}.$$

The following theorem describes the limiting behaviour of the sequence $e_{n,m}$ for $m$ fixed and $n \to \infty$.

THEOREM 2. *Assume that* (1.2) *holds for a number* $\eta$ *with* $\eta \neq \infty$ *in the way that* (i) $\eta$ *is not a root of unity and* (ii) $\eta$ *is a root of unity of order* $m_o$ *and satisfies* (1.3). *Then* (i) *for any* $m$ *and* (ii) *for any* $m$, $m \leqslant m_0 - 1$ *there holds*

$$e_{n,m}(u) \to H_\eta(u) + \sum_{k=0}^{m-1} \frac{\prod_{j=1}^{k} (1 - \eta^j)(-1)^{k+1} u^k}{B_k(u)\, B_{k+1}(u)} \qquad as \quad n \to \infty$$

*uniformly inside the domains described by Theorem* 1), *excluding* $\mathscr{B}$, *where* $\mathscr{B}$ *is the set of the zeros of the polynomials* $B_k = B_k(z, \eta)$, $k = 1, \ldots, m$.

Set $\delta(m, \eta) := min\{|z| : B_m(z, \eta) = 0, k = 1, \ldots, m\}$. From Theorem 2, we get

COROLLARY 1. *With the assumptions of Theorem* 2, *for any* $\varepsilon$, $0 < \varepsilon < 1$, *the Padé error function* $f - \pi_{n,m}$ *has for n sufficiently large not more than a finite number of zeros in* $0 < |z| < \delta(m, \eta)(1 - \varepsilon) \cdot |a_n/a_{n+1}|$.

The next results characterize the limiting behaviour of the error functions as $n \to \infty$, $m$-fixed, in the case when the numbers $\eta_n$ tend to $\eta = 1$ in the way described by (1.3).

Denote by $E_{n,m}(u)$ the error function $f - \pi_{n,m}$ normalized as follows:

$$E_{n,m}(u) := \frac{(f - \pi_{n,m})(ua_n/a_{n+1})}{(ua_n/a_{n+1})^{n+m+1} \cdot (-1)^m \cdot D(n+1, m+1)/D(n, m)}$$

In the present paper, we prove

THEOREM 3. *Let $m \in \mathbf{N}$ be fixed and $f \notin \mathscr{R}_m$. Assume that $a_j \neq 0$ for $j$ large; assume, further that $\eta_n$ admits the expansion (1.3) with $\eta = 1$, $c_1 \neq O$ and $|\eta_n| \leqslant 1$ for all $n \in \mathbf{N}$ sufficiently large.*

*Then*

$$E_{n,m}(u) \to (1-u)^{2m+1} \qquad as \quad n \to \infty$$

*uniformly inside $\{u : |u| < 1\}$.*

From Theorem 3, we have

COROLLARY 2. *With the assumptions of Theorem 3, for each fixed $m \in \mathbf{N}$ and any $\varepsilon$, $0 < \varepsilon < 1$, the Padé error function $(f - \pi_{n,m})(z)$ has no zeros in $0 < |z| < |a_n/a_{n+1}| \cdot (1 - \varepsilon)$ for $n$ sufficiently large.*

Recall that under our assumptions each Padé error function of order $(n, m)$ has a zero at $z = 0$ of order $m + n + 1$.

Further, we consider the special case when $\eta = 1$ and the first coefficient $c_1$ in (1.3) is a real negative number. Under this additional condition, the next result provides information about the existence of "extraneous" zeros of the normalized Padé error functions $E_{n,m}(u)$ and about the limiting behaviour of those zeros as $n \to \infty$, as well.

THEOREM 4. *If $\eta = 1$ and $c_1 < 0$, then $u = 1$ is a limit point of zeros of $\{E_{n,m}(u)\}_{n=1}^{\infty}$.*

For $n \in N$, we denote by $P_n$ the set of the zeros of $E_{n,m}(u)$. Set $P_n := \{\xi_{n,k}\}$ with the normalization $|1 - \xi_{n,k}| \leqslant |1 - \xi_{n,k+1}|$, $k = 1, \ldots$. From Theorem 4, we have

$$dist(P_n, 1) \to 0, \qquad \text{as} \quad n \to \infty.$$

For any positive $\varepsilon$, denote by $\iota_n(\varepsilon)$ the number of the zeros of $\xi_{n,k}$ which lie in the disk of radius $\varepsilon$, centered at $u = 1$. In the present paper we prove

THEOREM 5. *Under the conditions of Theorem 4, for for any* $\varepsilon$, *we have*

$$\liminf_{n\to\infty} \frac{l_n(\varepsilon)}{n} > 0. \tag{1.5}$$

For a number $\varepsilon$, $0<\varepsilon<1$, we denote by $\mathscr{A}_n(\varepsilon)$ the annulus $(1-\varepsilon)\,|a_n/a_{n+1}| < |z| < (1+\varepsilon)\,|a_n/a_{n+1}|$. From the last theorem, we have

COROLLARY 3. *If* $\eta=1$ *and* $c_1<0$, *then for any* $\varepsilon$, $0<\varepsilon<1$ *the Padé error function* $f-\pi_{n,m}$ *has, for* $n$ *large enough, extraneous zeros which are situated in the annulus* $\mathscr{A}_n(\varepsilon)$. *Their number* $k_n(\varepsilon)$ *satisfies, as* $n\to\infty$, *condition* (1.5).

Set

$$R := \liminf_{n\to\infty} |a_n/a_{n+1}|.$$

(Notice that $\rho(f)\geqslant R$.)

Obviously, if $R=\infty$, then $f$ is an entire function. If in addition the conditions of Theorem 4 are fulfilled, then each Padé error function $f-\pi_{n,m}$ has, for $n$ large enough, extraneous zeros and they go to infinity, as $n\to\infty$, with the speed of $|a_n/a_{n+1}|$.

Further, if $\eta=1$ and $0<R<\infty$, (in this case $\rho(f)>0$), then the set $\{z:|z|<R\}-0$ does not contain, in view of Theorem 3, accumulation points of the zeros of $(f-\pi_{n,m})$ as $n\to\infty$ (recall that each Padé error function has a zero at $z=0$ of order $m+n+1$). If in addition the parameter $c_1$ in (1.3) is a negative number, then, in accordance with Theorem 4, the circle $\{z:|z|=R\}$ contains accumulation points of zeros of $(f\in\pi_{n,m})(z)$ as $n\to\infty$. If $R=\limsup_{n\to\infty}|a_n/a_{n+1}|$, then $R=\rho(f)$ and all the extraneous zeros of $f-\pi_{n,m}$ tend to the circle $\{z:|z|=R\}$.

Finally, if $\rho(f)=0$, then $z=0$ is an accumulation point of extraneous zeros of $f-\pi_{n,m}$, as $n\to\infty$.

Important functions to which Theorem 4 may be applied are the exponential function (see [10])

$$f(z)=\exp z=\sum_{j=0}^{\infty} z^j/j!$$

and the Mittag-Lefler function of order $\lambda$, $\lambda>0$, (see [4])

$$f(z)=\sum_{j=0}^{\infty} z^j/\Gamma(1+j/\lambda), \qquad \lambda>0.$$

The Padé error function for $e^{-z}$ has been considered in [6].

## 2. Preliminaries

Lemma 1. *For any $n$ and $m$, there holds*

$$\frac{a_{n+m}}{a_n} = \left(\frac{a_{n+1}}{a_n}\right)^m \prod_{j=1}^{m-1} \eta_{n+m-j}^j.$$

The proof will be omitted.
Set

$$D_{n,m} := \frac{D(n, m)}{a_n^m}$$

The following lemma is of essential importance for all the considerations in the present

Lemma 2 (see [7]). *Let $f$ be a formal power series, with $a_j \neq O$ for $j$ large. Assume that $\eta j$ has the asymptotic expansion (1.3) with $c_1 \neq 0$. Then for $m = 1, 2, ...$ we have*

$$D_{n,m} = (-c_1/n)^{m(m-1)/2} \cdot \prod_{j=1}^{m-1} j^{m-j} \cdot \{1 + \alpha(1, m)/n + o(1/n)\} \qquad as \quad n \to \infty.$$

*and*

$$\lim_{n\to\infty} Q_{n,m}(ua_n/a_{n+1}) = B_m(u).$$

*If (1.2) holds for a number $\eta$ that is not a root of unity of order $m$ then*

$$\lim_{n\to\infty} D_{n,m} = \prod_{j=1}^{m-1} (1-\eta^j)^{m-j}.$$

*We set*

$$I_{n,j} := \prod_{l=0}^{j} \eta_{n+1+l}.$$

The next lemma describes the asymptotic behaviour of $I_{n,j}$ as $n \to \infty$ for $j$ "small". Before presenting it, we introduce for a given function $g$ and a fixed number $p$, $p \in \mathbf{N}$, the operator

$$\nabla^p g(x) := \sum_{i=0}^{p} (-1)^i \binom{p}{i} g(x-i),$$

with

$$\nabla^0 f(x) := f(x)$$

and

$$\nabla f(x) := \nabla^1 f(x)$$

LEMMA 3 (see [5]). *Assume that $\eta_j$ has the asymptotic expansion* (1, 3) *with $\eta = 1$ and $c_1 \neq 0$. Let $N$ be an arbitrary positive integer. Then*

(a) *for each $j$, $j+1 \leqslant n/N$ we have as $n \to \infty$*

$$I_{n,j} = 1 + \sum_{s=1}^{N} (j+1) \cdot Q_{s-1}(j)/(n+1)^s + M_{N+1}(j, n),$$

*where $Q_s$ is a polynomial of degree $\leqslant s$;*

(b) *for each fixed $p$, $p = 0, 1,...$ and for any $j$, $j+1 < (n-p)/(N+2p)$ the remainder $M_{N+1}(j, n)$ behaves according to*

$$(n+1)^{N+1+p} |\nabla^p M_{N+1}(j, n)| \leqslant \mathscr{C}_{N+1,p}(j+1)^{N+1}$$

*with suitable positive constants $C_{N+1,p}$ which do not depend on $n$ and $j$.*

We notice that

$$Q_0(j) = c_1 \tag{2.1}$$

and

$$Q_1(j) = (jc_1^2 - jc_1 + 2c_2)/2. \tag{2.2}$$

(In what follows, we shall denote by $C_{a,b,...}$ and $C(...)$ positive constants that do not depend on $n$.)

## 3. PROOFS OF THE RESULTS

*Proof of Theorem* 1. Applying Lemma 1 to the normalized error function $W_n(u)$, we easily get

$$W_n(u) = 1 + \sum_{j=1}^{\infty} d_{n,j} u^j,$$

where

$$d_{n,j} := \prod_{l=1}^{j} \eta_{n+j+1-l}^l. \tag{3.1}$$

First, we notice that under condition (1.2), for any fixed positive integer $k$ there holds

$$1+\sum_{j=1}^{k} d_{n,j}u^j \to \sum_{j=0}^{k} \eta_{j(j+1)/2} z^j \qquad \text{as} \quad n\to\infty. \tag{3.2}$$

We first consider case (i).

Suppose that $|\eta|<1$. Condition (1.2) ensures the existence of a positive number $\delta$ with

$$|\eta|<1-\delta. \tag{3.3}$$

Let $\varepsilon$ and $R$ be arbitrarily fixed positive numbers. Obviously, there is an integer $k$ such that

$$R(1-\delta)^{(k+1)/2}\leqslant 1/2 \tag{3.4}$$

and

$$(1/2)^k<\varepsilon/2. \tag{3.5}$$

We rewrite $W_n(u)$ as follows:

$$W_n(u)=1+\sum_{j=1}^{k-1} d_{n,j}u^j+\sum_{j=k}^{\infty} d_{n,j}u^j := W_{n,1,k}(u)+W_{n,2,k}(u)$$

From (3.3), we have for every $n$ large enough ($n>n_\delta$) the inequality

$$|d_{n,j}|\leqslant(1-\delta)^{j(j+1)/2}.$$

From the last inequality, we get for $n>n_\delta$

$$\|W_{n,2,k}(u)\|_{|u|\leqslant R}\leqslant(R(1-\delta)^{(k+1)/2})^k\cdot\sum_{l=0}^{\infty} R^l(1-\delta)^{(l^2+l(1+2k))/2}.$$

Making now use of (3.4) and (3.5), for every $n>n_\delta$ we obtain

$$\|W_{n,2,k}(u)\|_{|u|\leqslant R}<\varepsilon.$$

Together with (3.2), this inequality yields statement (i) of Theorem 1.

Assume now that the conditions of case (ii) hold; then for every $n$ large enough ($n>n_o$) we have

$$|d_{n,j}|\leqslant 1.$$

Repeating now the previous considerations, we come to statement (ii).

*Proof of Theorem* 2. From the equality (see [2])

$$(\pi_{n,k}-\pi_{n,k+1})(z)=(-1)^{k+1}\frac{z^{n+k+1}D(n+1),k+1)}{(Q_{n,k}Q_{n,k+1})(z)\,D(n,k)}$$

we get

$$\begin{aligned}(f-\pi_{n,m})(z)&=(f-S_n)(z)+\sum_{k=0}^{m-1}(\pi_{n,k}-\pi_{n,k+1})(z)\\ &=(f-S_n)(z)+\sum_{k=0}^{m-1}(-1)^{k+1}\\ &\quad\cdot\frac{D(n+1,k+1)}{D(n,k)}\cdot\frac{z^{n+k+1}}{Q_{n,k}(z)\,Q_{n,k+1}(z)},\end{aligned}$$

so that

$$e_{n,m}(u)=W_n(u)+\sum_{k=0}^{m-1}\frac{(-1)^{k+1}u^kD_{n+1,k+1}}{D_{n,k}\cdot(Q_{n,k}Q_{n,k+1})(ua_n/a_{n+1})}.$$

The statement of Theorem 2 follows now from Lemma 2 and from Theorem 1.

*Proof of Theorem* 3. Set

$$Q_{n,m}=\sum_{k=0}^{m}q_{k,n,m}z^{m-k}$$

(recall that $q_{m,n,m}=1$). Completing technical transformations we rewrite $E_{n,m}$ as

$$E_{n,m}(u)=\frac{1+\sum_{j=1}^{\infty}d_{n,j}F_{j,n,m}u^j}{Q_{n,m}(ua_n/a_{n+1}),}. \tag{4.1}$$

In the last formula

$$\begin{aligned}F_{j,n,m}=(-1)^m\frac{D_{n,m}}{D_{n+1,m+1}}\cdot\sum_{k=0}^{m}\frac{a_n^{m-k}}{a_{n+1}^{m-k}}\cdot q_{k,n,m}\\ \cdot\prod_{l=1}^{k}\eta^l_{n+k+j+1-l}\cdot\prod_{l=1}^{j}\eta^k_{n+j+1-l}.\end{aligned} \tag{4.2}$$

Indeed, from the definition of Padé approximants we have

$$\frac{(f\cdot Q_{n,m}-P_{n,m})(ua_n/a_{n+1})}{(ua_n/a_{n+1})^{n+m+1}}=\sum_{j=0}^{\infty}(ua_n/a_{n+1})^j\sum_{k=0}^{m}a_{n+k+j+1}q_{k,n,m}.$$

Applying now Lemma 1, we easily come to (4.1).

From Sylvester's identity (see [2]), there results the recurrence formulas

$$q_{k,n,m}=q_{k-1,n,m-1}-\tilde{D}(n,m)q_{k,n-1,m-1},$$

$$q_{0,n,m}=-\tilde{D}(n,m)q_{0,n-1,m-1},$$

where

$$\tilde{D}(n,m):=D(n-1,m-1)\,D(n+1,m)/D(n,m-1)\,D(n,m).$$

Making use of these formulas, we obtain

$$F_{j,n,m}:=\frac{-D(n,m)\,D(n+1,m)}{D(n+1,m+1)\,D(n,m-1)}\cdot\frac{a_{n+1}}{a_n}\cdot\{F_{j+1,n,m-1}\cdot I_{n,j}-F_{j+1,n-1,m-1}\}. \tag{4.3}$$

The next step in the proof is to establish by induction on $m$ that

(a) for every $N\in\mathbf{N}$ and for $j+m<n/(N+3m-1)$ the following expansion is valid as $n\to\infty$

$$F_{j,n,m}=\prod_{l=1}^{m}\frac{(j+l)}{m!}+\sum_{s=1}^{N}\frac{\mathscr{P}_{s,m}(j)}{(n+1)^s}+\mathscr{N}_{N+1,m}(j,n), \tag{4.4}$$

where $\mathscr{P}_{s,m}(j)$ is a polynomial of degree not exceeding $m+s$;

(b) for every $j$, satisfying $j+m<n/(N+1+m+2(p+m-1))$ there holds, as $n\to\infty$,

$$|(n+1)^{N+p+1}\cdot\nabla^p\mathscr{N}_{N+1,m}(j,n)|<C_{m},N+1,p\cdot j^{N+m+1} \tag{4.5}$$

with $C_{m,N+1,p}$ a positive constant not depending on $n$ and on $j$;

(c) for numbers $j$ with $j+m\geqslant n/3m$ we have for $n\to\infty$

$$|F_{j,n,m}|\leqslant \mathbf{c}_m\cdot j^m. \tag{4.6}$$

Check the hypothesis for $m=1$. The direct calculation gives

$$F_{j,n,1}=\frac{I_{n,j}-1}{\eta_{n+1}-1}. \tag{4.7}$$

In accordance with (1.3), we may write for $n$ sufficiently large

$$1/(\eta_{n+1}-1)=\left\{1+\sum_{i=1}^{N} g_i/(n+1)^i+o(1/n^N)\right\}\cdot(n+1)/c_1$$

with

$$g_1=-c_2/c_1.$$

Using Lemma 3, (2.1) and (4.7), we get for $F_{j,n,1}$

$$F_{j,n,1}=\left\{(j+1)\left\{1+\sum_{s=1}^{N}\frac{Q_s(j)}{c_1(n+1)^s}\right\}+\frac{(n+1)\cdot M_{N+2}(j,n)}{c_1}\right\}$$
$$\cdot\left\{1+\sum_{i=1}^{N+1}\frac{g_i}{(n+1)^i}+o(1/n^N)\right\}.$$

The last formula can be rewritten as

$$F_{j,n,1}=j+1+\sum_{s=1}^{N}\mathscr{P}_{s,1}(j)/(n+1)^s+\mathscr{N}_{N+1,1}(j,n).$$

We easily verify that $deg\,\mathscr{P}_{s,1}(j)\leqslant s+1$. Further, we see that $\mathscr{N}_{N+1,1}(j,n)$ depends on $(n+1)\cdot M_{N+2}(j,n)$. Let now $p\in\mathbf{N}$ be fixed. Since for $j+1<(n-p)/(N+1+2p)$ the remainder $M_{N+2}(j,n)$ behaves in the way described by Lemma 3, then $\mathscr{N}_{N+1,1}(j,n)$ satisfies as $n\to\infty$ the induction hypothesis (4.5) for numbers $j$ with $j+1<n/(N+2+2p)$ and $C_{1,N+1,p}$ being a suitable positive constant.

Also, for $j+1\geqslant n/3$, (4.7) implies (4.6) for $m=1$ with a suitable positive constant.

Thus the assertion is proved for $m=1$.

Set now

$$\mathscr{D}_{n,m}:=\frac{-D(n,m)\,D(n+1,m)}{D(n+1,m+1)\,D(n,m-1)}\cdot\frac{a_{n+1}}{a_n}.$$

Before proving the induction hypothesis for an arbitrary number $m$, we consider the asymptotic behaviour of $\mathscr{D}_{n,m}$ as $n\to\infty$ and $m$ is fixed. From Lemma 2, there follows for every $N\geqslant 1$ the representation

$$\mathscr{D}_{n,m}=\frac{n+1}{mc_1}\cdot\left\{1+\sum_{i=1}^{N}\frac{\alpha_{i,m}}{(n+1)^i}+o(1/(n+1)^N)\right\},\qquad \text{as}\quad n\to\infty.$$

Suppose (4.4)–(4.6) are valid for some $m$. Let $N \in \mathbf{N}$ be fixed. Then in view of (4.3) we may write

$$F_{j,n,m+1} = \frac{n+1}{(m+1)c_1}\left\{1+\sum_{i=1}^{N}\frac{\alpha_{i,m+1}}{(n+1)^i}+o(1/(n+1)^N)\right\}$$
$$\cdot\left\{\left\{1+\sum_{s=1}^{N+1}(j+1)\cdot Q_{s-1}(j)/(n+1)^s+M_{N+2}(j,n)\right\}\right.$$
$$\cdot\left\{\prod_{l=2}^{m+1}\frac{j+l}{m!}+\sum_{s=1}^{N+1}\frac{\mathcal{P}_{s,m}(j+1)}{(n+1)^s}+\mathcal{N}_{N+2,m}(j+1,n)\right\}$$
$$\left.-\left\{\prod_{l=2}^{m+1}\frac{j+l}{m!}+\sum_{s=1}^{N+1}\frac{\mathcal{P}_{s,m}(j+1)}{n^s}+\mathcal{N}_{N+2,m}(j+1,n-1)\right\}\right\}.$$

Using (2.1) and (2.2), we rewrite this formula in the required form, namely,

$$F_{j,n,m+1} = \prod_{l=1}^{m+1}\frac{(j+l)}{(m+1)!}+\sum_{s=1}^{N}\mathcal{P}_{s,m+1}(j)/(n+1)^s+\mathcal{N}_{N+1,m+1}(j,n),$$

where obviously each polynomial $\mathcal{P}_{s,m+1}$ is of degree not exceeding $m+s+1$. Further, we see that the remainder $\mathcal{N}_{N+1,m+1}(j,n)$ depends on the difference $(n+1)\cdot\{\mathcal{N}_{N+2,m}(j+1,n)-\mathcal{N}_{N+2,m}(j+1,\ n-1)\} = (n+1)\cdot\nabla\mathcal{N}_{N+2,m}(j+1,n)$. From the definition of $\nabla^p$ we easily set

$$\nabla^p\{(n+1)\cdot\nabla\mathcal{N}_{N+2,m}(j+1,n)\}$$
$$=(n+1)\cdot\nabla^{p+1}\mathcal{N}_{N+2,m}(j+1,n)-p\cdot\nabla^p\mathcal{N}_{N+2,m}(j+1,n-1).$$

Therefore, for $j+1+m<\mu_{n,p}:=min(n/(N+2+m+2(p+m)),\ (n-1)/(N+2+m+2(p+m-1)))$ the term $\nabla^p\mathcal{N}_{N+1,m+1}(j,n)$ behaves in the way described by (4.6). The observation that $\mu_{n,p}=n/(N+2+m+2(p+m))$ for $n$ large establishes (4.5) for $m+1$.

Further, for $j+1+m\geqslant n/(3m+3)$ we easily check that

$$|F_{j,n,m+1}|\leqslant \mathbf{c}_{m+1}\,j^{m+1}.$$

This proves the induction hypothesis (4.4)–(4.6) for $m+1$.

We notice for $j+m<n/3m$ the validity of the inequalities

$$F_{j,n,m}=\prod_{l=1}^{m}\frac{(j+l)}{m!}+\mathcal{N}_{1,m}(j,n) \tag{4.8}$$

with

$$|n\cdot\mathcal{N}_{1,m}(j,n)|<\mathcal{C}_{1,m}(j,n)|<\mathcal{C}_m\cdot j^{m+1}\qquad\text{as}\quad n\to\infty. \tag{4.9}$$

We now are in position to prove Theorem 2.

Set, for any $k \in \mathbf{N}$

$$e_{n,m,k}(u) := \sum_{j=0}^{k-1} d_{n,j} F_{j,n,m} u^j$$

and

$$\mathscr{E}_{n,m,k}(u) := \sum_{j=k}^{\infty} d_{n,j} F_{j,n,m} u^j.$$

From (4.1), we have

$$E_{n,m}(u) = \frac{e_{n,m,k}(u) + \mathscr{E}_{n,m,k}(u)}{Q_{n,m}(ua_n/a_{n+1})}. \tag{4.1$'$}$$

Let $\delta$ be a fixed positive number. For $n \in \mathbf{N}$ sufficiently large we write

$$\mathscr{E}_{n,m,k}(e^{-\delta}) := \left\{ \sum_{j=k}^{j<n/3m-m} + \sum_{j \geqslant n/3m-m}^{\infty} \right\} d_{n,j} F_{j,n,m} e^{-\delta j}.$$

Applying (4.4)–(4.6), we establish that

$$\|\mathscr{E}_{n,m,k}\|_{|u| \leqslant e^{-\delta}} \leqslant C(e^{-k\delta} + e^{-n\delta}),$$

where $C$ is a constant independent of $k$ and $n$. On the other hand, for any fixed $j$ we have

$$F_{j,n,m} \to \prod_{l=1}^{m} \frac{(j+l)}{m!} \qquad \text{as} \quad n \to \infty,$$

so that, for any fixed $k$ we may write

$$e_{n,m,k}(u) \to \sum_{j=0}^{k-1} \prod_{l=1}^{m} \frac{(j+l)}{m!} \qquad \text{as} \quad n \to \infty.$$

Hence the statement of Theorem 3 follows easily. Indeed,

$$\sum_{j=0}^{\infty} u^j \prod_{l=1}^{m} (j+l)/m! = \frac{1}{m!} \cdot (u^m(1-u)^{-1})^{(m)} = (1-u)^{-m-1}.$$

This result, (4.1)$'$, and Lemma 2 establish the statement of Theorem 3.

*Proof of Theorem* 4. In our further considerations, we shall deal with the functions $\mathscr{E}_{n,m}(u) := \mathscr{E}_{n,m,0}(u)$. Recall that $m$ is fixed and $n \to \infty$. In accordance with (3.1) we have

$$d_{n,j} := \prod_{l=1}^{j} \eta^l_{n+j+1-l} \qquad \text{as} \quad n \to \infty.$$

Arguing in the same way as in [5], we shall establish that for every $\delta$ small enough there exists a positive integer $n_\delta$ such that for any $n > n_\delta$ the inequality

$$\operatorname{Re} \mathscr{E}_{n,m}(e^{2\delta}) \geqslant C(0) \cdot e^{n \cdot \alpha(\delta)} \tag{5.1}$$

is valid, where

$$\alpha(\delta) := \delta^2/(-c_1 + \delta)$$

and $C(0)$ is a positive constant, independent of $n$ and $\delta$.

For convenience, we shall use the notation $c_1 := -2d_1$. In the conditions of Theorem 4, $d_1 > 0$. In [5], the validity of the following inequalities for each $n$ large $(n > n_0)$ was established:

$$|\eta_n| \leqslant 1 - d_1/n \tag{5.2}$$

and

$$|\eta_n| \leqslant |\eta_{n+1}|. \tag{5.3}$$

The two last iunequalities lead to

$$|d_{n,j}| \leqslant (|\eta_{n+j}|)^{j(j+1)/2} \leqslant (1 - d_1/(n+j))^{j(j+1)/2}. \tag{5.4}$$

This inequality means that each function $\mathscr{E}_{n,m}(u)$ admits an analytic continuation in the disk $\{u, |u| < e^{d_1/2}\}$.

Let $\varepsilon$ be a fixed positive number, $\varepsilon < 1$.

In our further considerations, we will assume that for $n > n_0$ the inequalities

$$(n/d_1)|\log(1 - d_1/n)| \geqslant 1 - \varepsilon \tag{5.5}$$

and

$$|\operatorname{Im} \eta_n| \leqslant C(1) \cdot \operatorname{Re} \eta_n/n^2 \tag{5.6}$$

are fulfilled for a suitable positive constant $C(1)$. Without loss of generality, we may assume that $C(1) > 1$. In accordance with Theorem 2, (4.8), (4.9) and (4.6), we may also write

$$|\mathscr{N}_{1,m}(j, n)| \leqslant C(1)) \cdot j^{m+1}/n \tag{5.7}$$

for $j + m < n/3m$ and

$$|F_{j,n,m}| \leqslant C(1)) j^m \tag{5.8}$$

otherwise. Select a positive number $\delta_0$ such that

$$0 < 6C(1)m!\delta_0/(d_1(1-\varepsilon)-6\delta_0) < 1/3, \tag{5.9}$$

and set

$$d(\varepsilon, \delta_0) := d_1(1-\varepsilon)-6\delta_0.$$

In what follows, we shall assume that each $n > n_0$ satisfies the inequality

$$\operatorname{Re}\eta_n \geqslant 1-(2d_1+\delta_0)/n > 0. \tag{5.10}$$

Let $\delta$ be a positive number such that $\delta < \delta_0$. Obviously, there is an integer $n_\delta$, $n_\delta > n_0$, such that for any $n \geqslant n_\delta$ the inequalities

$$\operatorname{Re}\eta_n \geqslant 1-(2d_1+\delta)/n \tag{5.11}$$

and

$$\frac{(n+1)}{2d_1+\delta}\cdot\left|\log\left(1-\frac{2d_1+\delta}{n+1}\right)\right| < 2 \tag{5.12}$$

are fulfilled. Set $j_1(\delta) := 6\delta/d(\varepsilon, \delta)$. For $j > j_1(\delta)\cdot n$ we obtain, in accordance with (5.4) and (5.5) that

$$|d_{n,j}| \leqslant e^{-3j\delta},$$

which implies with respect to (5.9), (5.7) and to the choice of $\delta$ the inequality

$$\left\|\sum_{j=j_1(\delta)n}^{\infty} d_{n,j}F_{j,n,m}u^j\right\|_{|u|=e^{2\delta}} \leqslant C(\delta_0)e^{-\delta j_1(\delta)n}. \tag{5.13}$$

On the other hand, we have for $j+1 \leqslant j_1(\delta)\cdot n$ by (5.3), (5.6) and the choice of $\delta$

$$\left|\frac{d_{j,n}}{\prod_{l=1}^{j}(\operatorname{Re}\eta_{n+j}+1-l)^l}-1\right| \leqslant C(2)\cdot\delta^2,$$

where $C(2)=18C(1)/d^2_{\varepsilon,\delta}$. Notice that the choice of $\delta$ ensures that $C(2)\cdot\delta_0^2 < 1/2$.

Further, the last inequality with respect to $d_{n,j}$ leads to

$$(1-C(2)\delta^2)\prod_{l=1}^{j}(\operatorname{Re}\eta_{n+j+1-l})^l \leqslant \operatorname{Re} d_{n,j}$$

$$\leqslant (1+C(2)\delta^2)\prod_{l=1}^{j}(\operatorname{Re}\eta_{n+j+1-l})^l$$

and

$$|\operatorname{Im} d_{n,j}| \leqslant C(2)\delta^2 \prod_{l=1}^{j} (\operatorname{Re} \eta_{n+j+1-l})^l.$$

Next, we assume without loss of generality that the integer $n_\delta$ satisfies the additional condition

$$j_1(\delta) \cdot n_\delta + m - 1 < n_\delta/3m. \tag{5.14}$$

Using now (4.8), (5.7), and the last inequalities, we get

$$\operatorname{Re} F_{j,n,m} \cdot \operatorname{Re} d_{n,j} - \operatorname{Im} F_{j,n,m} \cdot \operatorname{Im} d_{n,j} \geqslant Q_{\delta_0}(j) \cdot \prod_{l=1}^{j} (\operatorname{Re} \eta_{n+j+1-l})^l,$$

with

$$Q_{\delta_0}(j) := (1 - C(2)\delta_0^2) \cdot \left( \prod_{l=1}^{m} (j+l)/m! - C(1) j^m \cdot \frac{\delta_0}{d(\varepsilon, \delta_0)} \right)$$
$$- C(2)\, \delta_0^2 C(1)\, j^m \cdot \frac{6\delta_0}{d(\varepsilon, \delta_0)}.$$

As we see, $Q_{\delta_0}$ is a polynomial of degree exactly $m$ and all its coefficients are positive. Further, in view of (5.10) and to the choice of $\delta_0$ we may write

$$\operatorname{Re} F_{j,n,m} \cdot \operatorname{Re} d_{n,j} - \operatorname{Im} F_{j,n,m} \cdot \operatorname{Im} d_{n,j} > 0. \tag{5.15}$$

Recall that the last inequality is valid for $n > n_\delta$ (compare with (5.14)) for any $j$ with $j+1 < j_1(\delta) \cdot n$. Set now $j_2(\delta) := \delta/(2d_1 + \delta)$ and consider $\mathscr{E}_{n,m,\delta}(e^{2\delta}) := \sum_{j=0}^{j_2(\delta)(n+1)-1} d_{n,j} F_{j,n,m}(e^{2\delta j})$. From (5.12) we get

$$j+1 < \frac{2\delta}{\left| \log \left( 1 - \dfrac{2d_1 + \delta}{n+1} \right) \right|},$$

which immediately implies

$$\left( 1 - \frac{2d_1 + \delta}{n+1} \right)^{j(j+1)/2} > e^{-\delta j}.$$

Taking into account (5.11), we conclude from this inequality that

$$\prod_{l=1}^{j} (\operatorname{Re} \eta_{n+j+1-l})^l > e^{-\delta j}.$$

Recall that the last inequality holds for $j < j_2(\delta)(n+1) - 1$ and for $n$ "large". Now, combining (4.13), (5.15), and the last result, we get

$$\operatorname{Re} \mathscr{E}_{n,m}(e^{2\delta}) > \sum_{j=0}^{j_2(\delta)(n+1)-1} Q_{\delta_0}(j) e^{\delta j} - C(\delta_0) e^{-6(\delta)^2 n/d(\varepsilon,\delta)}.$$

Inequality (5.1) results from here.

Now, it easily follows that the point $u = 1$ attracts, as $n \to \infty$, at least one zero of the sequence $E_{n,m}(u)$. Before presenting the proof, we introduce the notation $U_a(r)$; that is the open disk of radius $r$, centered at the point $a$; further, we set $\Gamma_a(r) := \partial U_a(r)$.

Assume now the contrary that there is a disk $U_1(e^{-\rho})$, $1 + e^{-\rho} < e^{d_1/2}$ such that $\mathscr{E}_{n,m}(u) \neq 0$ there. Set $\tau := \log(1 + e^{-\rho})$. Let $\theta$ be an arbitrary positive number with $1 + e^{-\rho} \cdot e^{\theta} < e^{d_1/2}$ and $0 < 1 - e^{-\rho} \cdot e^{\theta}$. Set $\tau(\theta) := \log(1 + e^{-\rho + \theta})$. Without loss of generality we may assume that the number $\tau(\theta)/2$ satisfies inequality (5.9). In the notations of the preceding considerations, we introduce for $n$ "large" in the previous sense that series $A_n$ and $B_n$ as follows:

$$A_n(u) := \sum_{j+1 > j_1(\tau(\theta)/2) \cdot n}^{\infty} d_{n,j} F_{j,n,m} u^j$$

and

$$B_n(u) := \sum_{j=0}^{j_1(\tau(\theta)/2) \cdot n - 1} d_{n,j} F_{j,n,m} u^j.$$

Repeating the same considerations as above (see (5.13)), we establish that

$$\|A_n\|_{U_0(e^{\tau(\theta)})} \leqslant C(\tau(\theta)) e^{-n\tau(\theta) j_1(\tau(\theta)/2)}. \tag{5.16}$$

On the other hand, for $B_n$ we easily get

$$\|B_n\|_{U_0(e^{\tau(\theta)})} \leqslant C(\tau(\theta)) e^{n\tau(\theta) j_1(\tau(\theta)/2)}. \tag{5.17}$$

Set now $V := U_0(1) \cup U_1(e^{-\rho})$ and let $X_n$ be the regular branch of $(\mathscr{E}_{n,m})(u)^{1/n}$ determined by the condition $X_n(0) = 1$. Inequalities (5.16) and (5.17) ensure the uniform boundedness of the sequence $\{X_n\}$ in $V$. By Theorem 2 and by the theorem of uniqueness for holomorphic functions,

$$X_n \to 1$$

uniformly inside $U_0(1)$ and therefore, inside $V$, as well. On the other hand, by (5.1) for any $\delta < \tau$ the inequality $X_n(e^{\delta}) \geqslant \exp n(\delta^2/2(4d_1 + \delta))$ is valid for each $n$ large enough. This contradicts with the last result about the convergence of the sequence $X_n(u)$. The contradiction we obtained establishes the statement of Theorem 5.

*Proof of Theorem* 5. Preserving the notations of Theorem 4, denote now by $\xi_{n,k}$, $k = 1, ..., \iota_n$ the zeros of $\mathscr{E}_{n,m}(u)$ in $U_1(e^{-\rho})$. By Theorem 3, $\iota_n \geqslant 1$.

We shall show that

$$\liminf_{n\to\infty} \iota_n/n > 0.$$

Suppose to the contrary that there is an infinite sequence $\Lambda$, $\Lambda \subset \mathbf{N}$ such that

$$\lim_{n\to\infty,\, n\in\Lambda} \iota_n/n = 0. \tag{6.1}$$

Set

$$q_n(u) := \prod_{k=1}^{\iota_n} \left(1 - \frac{u}{\xi_{n,k}}\right)$$

and

$$\chi_n(u) := \left\{\frac{\mathscr{E}_{n,m}(u)}{q_n(u)}\right\}^{1/n},$$

with $\chi_n(0) = 1$.

Consider the sequence $\{\chi_n\}_{n\in\Lambda}$.

For $q_n(u)$ we have

$$\min_{u\in\Gamma_0(e^{\tau(\theta)})} |q_n(u)| \geqslant \left\{\frac{e^{-\rho}(e^{\theta}-1)}{(1+e^{-\rho})}\right\}^{\iota_n}.$$

Combine now the last equality, (5.16) and (5.17). By virtue of (6.1) and by the maximum principle for holomorphic functions, the sequence $\{\chi_n\}_{n\in\Lambda}$ is uniformly bounded on $V$. (recall that accordingly to the geometric construction and to the choice of $\theta$, $V \subset U_0(\tau(\theta))$).

Select now a positive number $r$ with $r < 1 - e^{-\rho}e^{\theta}$. For $u \in U_0(r)$, we obviously have

$$\left\{\frac{(1-e^{-\rho}+r)}{(1-e^{-\rho}}\right\}^{\iota_n} \geqslant |q_n(u)| \geqslant \left\{\frac{(1-e^{-\rho}-r)}{(1-e^{-\rho})}\right\}^{\iota_n}.$$

Therefore, in view of Theorem 4 and of (6.2), we may write

$$\chi_n \to 1 \qquad \text{as} \quad n \to \infty, \quad n \in \Lambda,$$

on the disk $U_0(r)$. Therefore

$$\chi_n \to 1 \qquad \text{as} \quad n \to \infty, \quad n \in \Lambda, \tag{6.2}$$

uniformly inside the domain $V$.

Select a positive number $\varepsilon_0$ such that

$$\varepsilon_0 < \frac{e^{-\rho}}{4}.$$

Set $\Omega(\varepsilon_0) := \bigcup_{n \in \Lambda} \bigcup_{k=1}^{t_n} \{u, |u - \xi_{n,k}| < \varepsilon_0 / t_n \cdot n^2\}$. Obviously,

$$mes_1(\Omega(\varepsilon_0)) < \varepsilon_0 < \frac{e^{-\rho}}{4}. \tag{6.3}$$

Further, for $u \in U_1(e^{-\rho}) - \Omega(\varepsilon_0)$ we have

$$\left\{\frac{2e^{-\rho}}{(1 - e^{-\rho})}\right\}^{t_n} \geqslant |q_n(u)|.$$

The choice of $\varepsilon_0$ and (6.3) ensures the existence of a positive number $\delta$, $\delta < \tau$ such that $e^{\delta} \in U_1(e^{-\rho}) - \Omega(\varepsilon_0)$. Applying (5.1) to that number $\delta$, using the last estimate and (6.1), we conclude that $\chi_n(e^{\delta}) > e^{n\delta^2/2(4d_1 + \delta)}$ for $n$ large enough. This inequality forms a contradiction with (6.2). Consequently, (1.5) is valid and Theorem 5 is true.

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